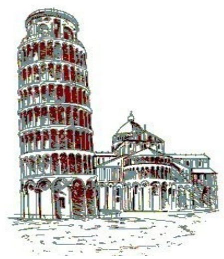
Corpi rigidi
summary

1) $L = L_{BO} + L_r$

2) $$\vec{L}_{cm} = \sum m_i \vec{r}_{bi} \wedge \vec{v}_i = \sum m_i \vec{r}_{bi} \wedge \vec{v}_b + \sum m_i \vec{r}_{bi} \wedge \vec{v}_{bi} = \vec{L}_r$$

3) $$\vec{M}_O = \vec{r}_b \wedge \vec{F}^E + \vec{M}_B^E$$

4) $$\vec{M}_{cm} = \sum \vec{r}_{bi} \wedge \vec{F}_i = \sum \vec{r}_{bi} \wedge (\vec{F}_{ib} + m_i \vec{a}_b) = \sum \vec{r}_{bi} \wedge m_i \vec{a}_b + \sum \vec{r}_{bi} \wedge \vec{F}_{ib} = \sum \vec{r}_{bi} \wedge \vec{F}_{ib}$$



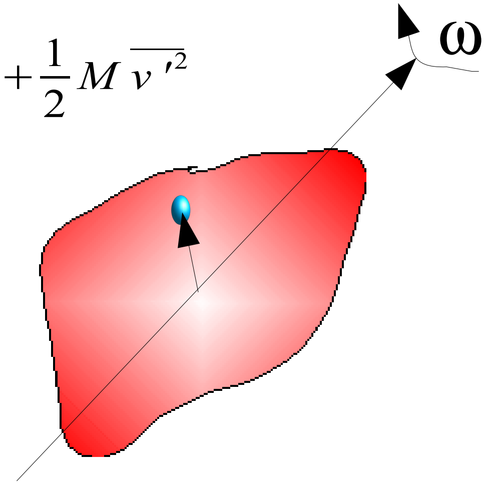
Corpi rigidi

Energia cinetica

$$T=\frac{1}{2}\int_M v^2\,dm=\frac{1}{2}\int_M (\vec{v}_b+\vec{v}\,')^2\,dm=\frac{1}{2}\int_M (v_b^2+2\,\vec{v}_b\cdot\vec{v}\,'+v'^2)\,dm$$

$$T=\frac{1}{2}M\,\overline{(v_b^2+2\,\vec{v}_b\cdot\vec{v}\,'+v'^2)}=\frac{1}{2}M\,(\overline{v_b^2}+\overline{2\,\vec{v}_b\cdot\vec{v}\,'}+\overline{v'^2})=\frac{1}{2}M\,v_b^2+\frac{1}{2}M\,\overline{v'^2}$$

$T = T_b + T'$ Köning



$$T'=\frac{1}{2}\int_M (\vec{\omega}\wedge\vec{r})^2\,dm=\frac{\omega^2}{2}\int_M d^2\,dm=\frac{1}{2}I_\omega\,\omega^2$$

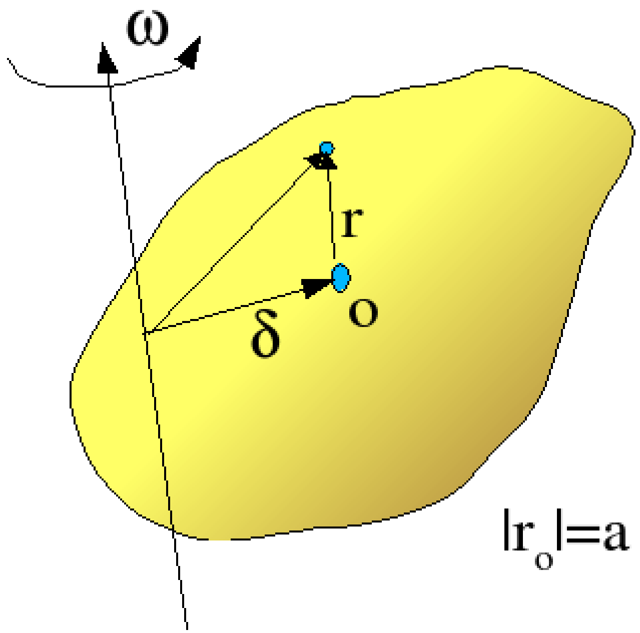
Momento di inerzia assiale

Corpi rigidi
energia cinetica

$$T' = \frac{1}{2}\int_M (\vec{\omega}\wedge(\vec{\delta}+\vec{r}))^2\, dm = \frac{\omega^2}{2}\int_M \delta^2\, dm + \frac{\omega^2}{2}\int_M d^2\, dm = \frac{1}{2} M\,\delta^2\,\omega^2 + \frac{1}{2} I_\omega \omega^2 = \frac{1}{2} I\,\omega^2$$

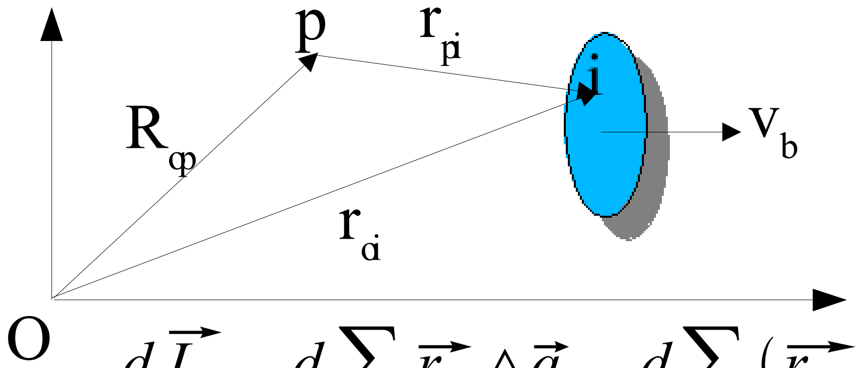
$$I = I_\omega + M\,\delta^2$$

Huygens Steiner



Corpi rigidi

polo mobile



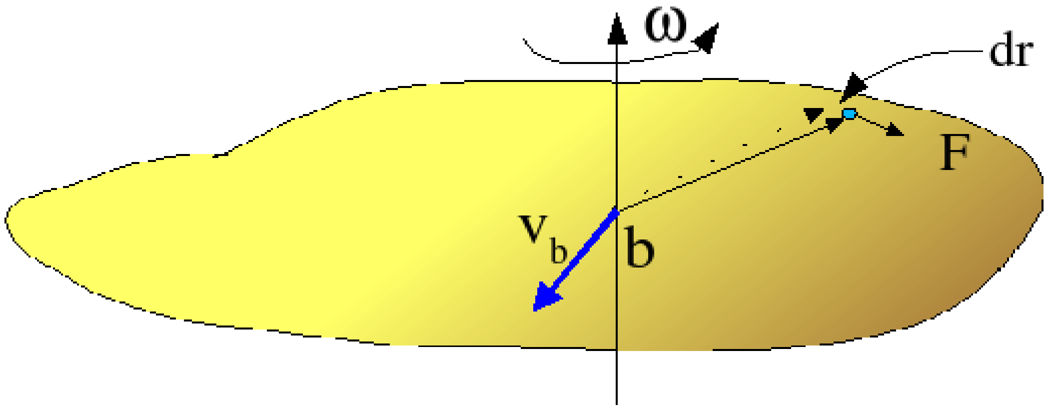
$$\frac{d\vec{L}_p}{dt} = \frac{d\sum \vec{r}_{pi}\wedge\vec{q}_i}{dt} = \frac{d\sum(\vec{r}_{pO}+\vec{r}_{Oi})\wedge\vec{q}_i}{dt} = \frac{d\sum(-\vec{r}_{Op}+\vec{r}_{Oi})\wedge\vec{q}_i}{dt}$$

$$\frac{d\vec{L}_p}{dt} = \sum(\vec{v}_i-\vec{v}_p)\wedge\vec{q}_i + \sum\vec{r}_{pi}\wedge\dot{\vec{q}}_i = -\vec{v}_b\wedge\vec{Q}+\sum\vec{r}_{pi}\wedge F_i$$

$$\dot{\vec{L}}_p = \vec{M}_p - \vec{v}_p\wedge\vec{Q} \qquad \textit{classica se} \quad \vec{Q}=0 \quad \textit{o se} \quad \vec{Q}\parallel\vec{v}_b$$

Corpi rigidi

Il lavoro



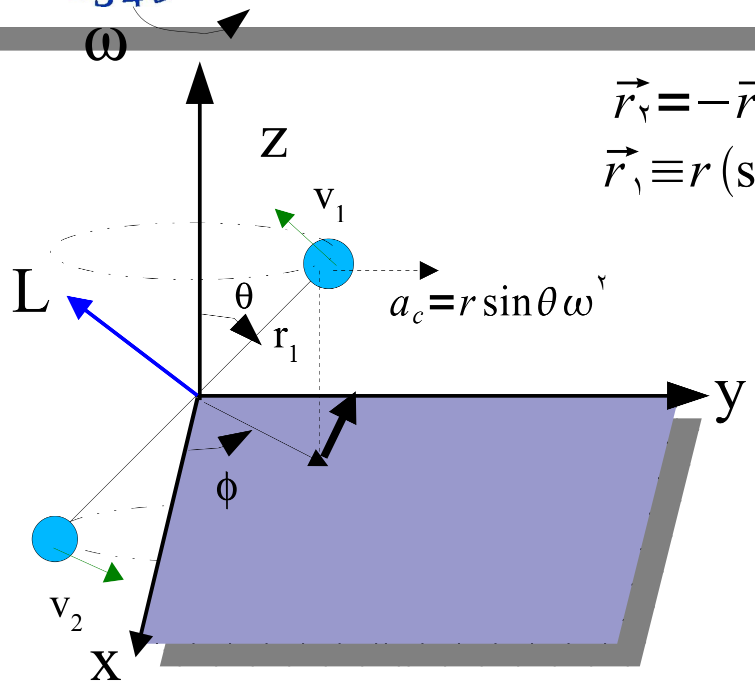
$$dL=\int \vec{f}(xyz)\cdot d\vec{r}\,dV=\int \vec{f}\cdot(\vec{v}_b+\vec{\omega}\wedge\vec{r})\,dt\,dV=\int \vec{f}\cdot\vec{v}_b\,dt\,dV+\int \vec{f}\cdot\vec{\omega}\wedge\vec{r}\,dt\,dV$$

$$\vec{F}^e\cdot\vec{v}_b\,dt+\int \vec{r}\wedge\vec{f}\,dV\cdot\vec{\omega}\,dt=\vec{F}^e\cdot\vec{v}_b\,dt+\vec{M}^e\cdot\vec{\omega}\,dt=\vec{F}^e\cdot d\vec{r}+\vec{M}^e\cdot\vec{d}\phi$$

Le forze interne non fanno lavoro!

Corpi rigidi

Due masse



$$\vec{r}_{٢} = -\vec{r}_{١} \quad \phi = \omega t \quad \vec{v}_{١} = \vec{r}_{١} \wedge \vec{\omega} \quad \vec{v}_{٢} = -\vec{v}_{١}$$

$$\vec{r}_{١} \equiv r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \quad \vec{\omega} \equiv \omega(٠, ٠, ١)$$

$$\vec{v}_{١} \equiv \omega r(-\sin\theta\sin\phi, \sin\theta\cos\phi, ٠)$$

$$\vec{L} = \vec{r}_{١} \wedge m\vec{v}_{١} + \vec{r}_{٢} \wedge m\vec{v}_{٢} = ٢(\vec{r}_{١} \wedge m\vec{v}_{١})$$

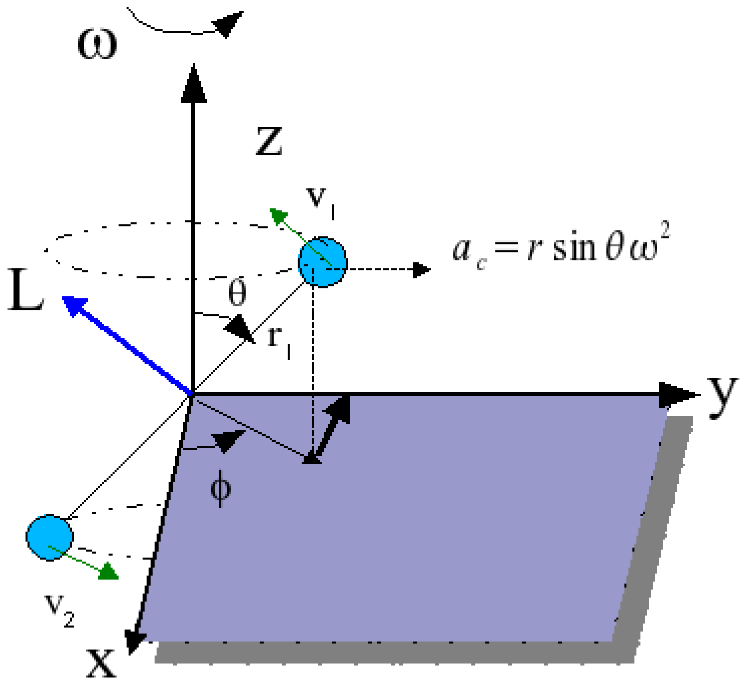
$$\vec{L} = ٢m(r^{٢}\vec{\omega} - (\vec{\omega}\cdot\vec{r}).\vec{r})$$

$$\vec{L} \equiv ٢mr^{٢}\omega(-\cos\theta\sin\theta\cos\phi, -\cos\theta\sin\theta\sin\phi, \sin\theta^{٢}) \quad not \parallel \vec{\omega} \quad ???$$

Corpi rigidi

Due masse

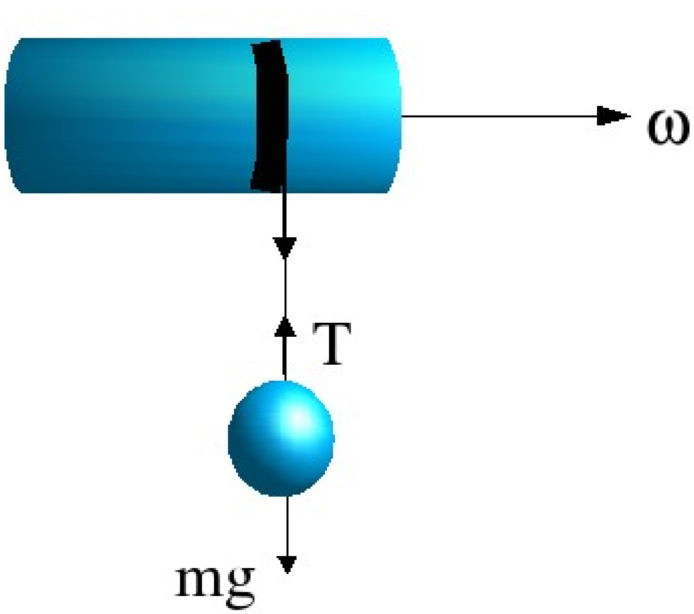
$$\vec{L} \equiv 2mr^2\omega(-\cos\theta\sin\theta\cos\phi, -\cos\theta\sin\theta\sin\phi, \sin\theta^2) \quad not \parallel \vec{\omega} \quad ???$$



$$\dot{\vec{L}} = \vec{M} \equiv m r^2 \omega^2 \cos\theta(\sin\phi, -\cos\phi, 0)$$

Corpi rigidi

Carrucola



$$\dot{L} = I\dot{\omega} = T\,R$$

$m\ddot{z} = mg - T$ con $\ddot{z} = R\dot{\omega}$ si ha

$T = mg - mR\dot{\omega}$ segue

$$I\dot{\omega} = mgR - m\,R^2\,\dot{\omega}$$

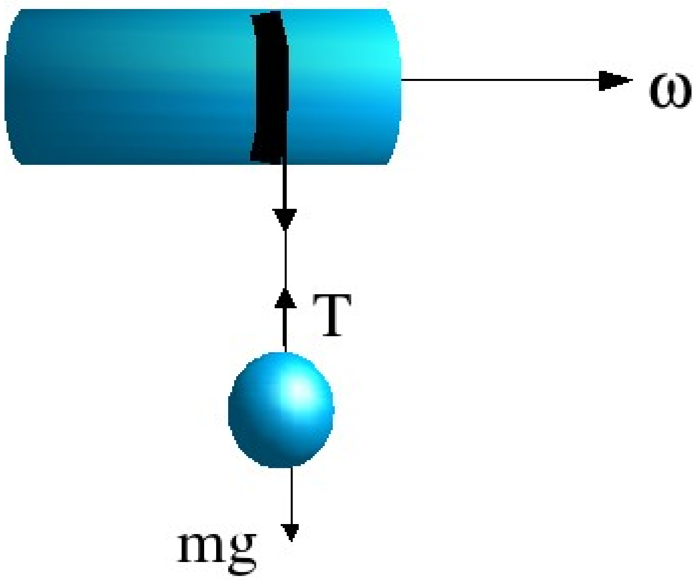
$$\dot{\omega} = \frac{mgR}{I + m\,R^2}$$

$$\dot{\omega} = \frac{mgR}{\frac{1}{2}M\,R^2 + m\,R^2} = \frac{1}{R}\,\frac{2mg}{M + 2m} \qquad a = R\,\dot{\omega} = \frac{2m}{M + 2m}\,g$$

Corpi rigidi

Carrucola

$$E = \frac{1}{2} I \omega^2 + \frac{1}{2} m v^2 + m g h$$



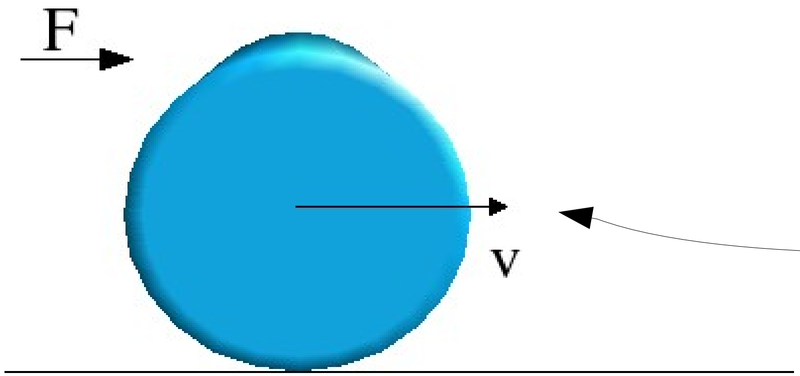
$$I \dot{\omega} \omega + m R^2 \dot{\omega} \omega + m g R \omega = 0$$

$$\dot{\omega} = \frac{m g R}{I + m R^2}$$

Corpi rigidi

cilindro che rotola

$$\dot{\vec{Q}} = \vec{F} + \vec{F}_r$$



$$\dot{L} = I\dot{\omega} = (I_b + M R^2)\dot{\omega} = R F$$

$$da cui \quad \omega = \frac{RF}{I} t + cost.$$

$$v = R\omega = \frac{R^2}{I_b + M R^2} F t = \frac{2}{3}\frac{F}{M} t$$

$$T = \frac{1}{2} I_b \omega^2 + \frac{1}{2} M v^2 = \frac{1}{2}(I_b + M R^2)\omega^2$$

Forze vive

$$dT = (I_b + M R^2)\dot{\omega}\,\omega\, dt = F\, ds = F\, R\omega\, dt$$

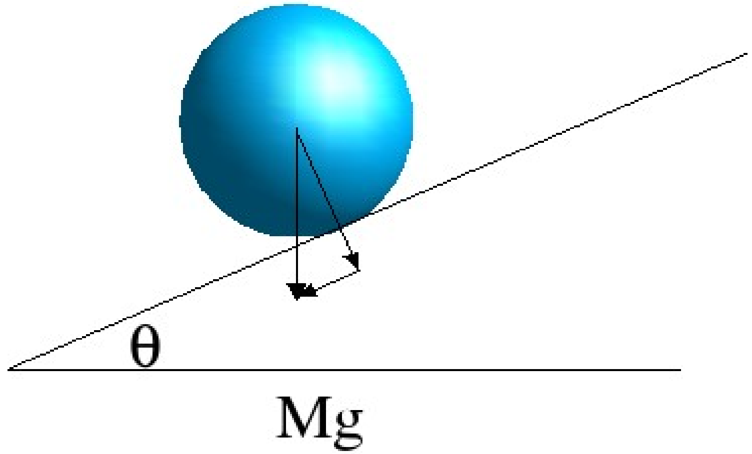
$$(I_b + M R^2)\dot{\omega} = FR$$

Corpi rigidi

cilindro che rotola

$$(I_b + M R^2)\dot{\omega} = \frac{7}{5} M R^2 \dot{\omega} = MgR\sin\theta$$

$$a = R\dot{\omega} = \frac{5}{7} g \sin\theta$$



$$T = \frac{١}{٢} I_s \omega^٢ + \frac{١}{٢} M v^٢ = \frac{١}{٢}\frac{٧}{٥} M R^٢ \omega^٢$$

$$E = \frac{٧}{١٠} MR^٢ \omega^٢ + Mgz\sin\theta \quad derivando\,con \;\; \dot{z} = R\omega$$

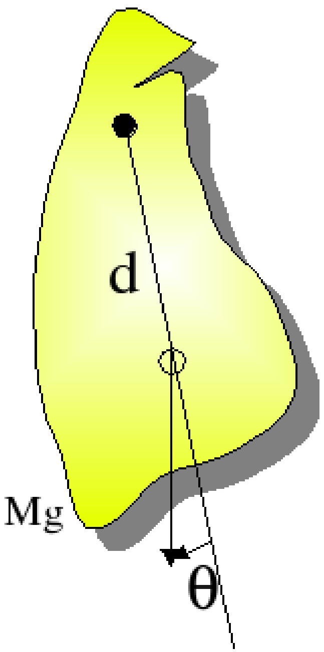
$$\frac{٧}{٥} MR^٢ \dot{\omega} + MgR\sin\theta = ٠$$

Corpi rigidi

pendolo composto

$$\dot{L} = I_n \ddot{\theta} = -Mgd\sin\theta \qquad \omega = \sqrt{\frac{Mgd}{I_n}} \equiv \sqrt{\frac{g}{l}}$$

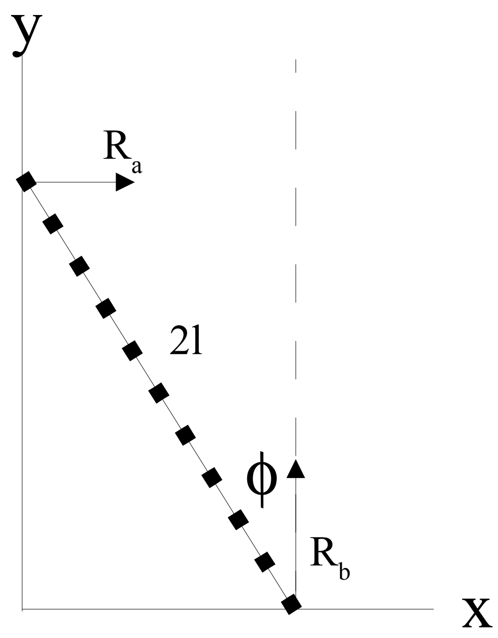
$$I_n = I_b + Md^2$$



$$l = \frac{I_n}{Md} = d + \frac{I_b}{Md}$$

$$x = l\sin\phi \qquad \dot{x} = +l\,\dot{\phi}\cos\phi \qquad \ddot{x} = l\,\ddot{\phi}\cos\phi - l\,\dot{\phi}^2\sin\phi$$

$$y = l\cos\phi \qquad \dot{y} = -l\,\dot{\phi}\sin\phi \qquad \ddot{y} = l\,\ddot{\phi}\sin\phi - l\,\dot{\phi}^2\cos\phi$$



$$\dot{\vec{Q}} = M\,\dot{\vec{v}}_b = \vec{R}_a + \vec{R}_b + m\,\vec{g}$$

$$M\,\ddot{x} = M\,(l\,\ddot{\phi}\cos\phi - l\,\dot{\phi}^2\sin\phi) = \quad -R_a$$

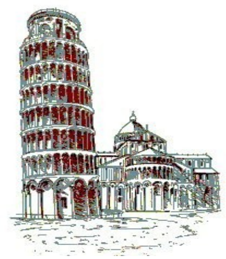
$$M\,\ddot{y} = M\,(l\,\ddot{\phi}\sin\phi - l\,\dot{\phi}^2\cos\phi) = R_b - Mg$$

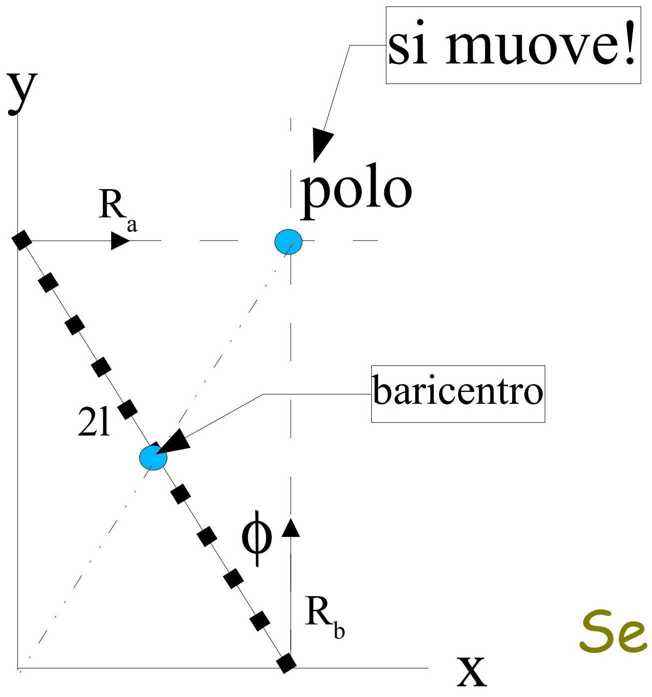
$$\vec{L}_b = I\,\dot{\vec{\phi}} \qquad \textit{dove } I \equiv I_b \textit{ rispetto asse } z$$

$$I_b\,\ddot{\phi} = -l\,R_a\cos\phi + l\,R_b\sin\phi$$

Corpi rigidi

scala





$$(I_b + M l^2)\dot{\phi}$$

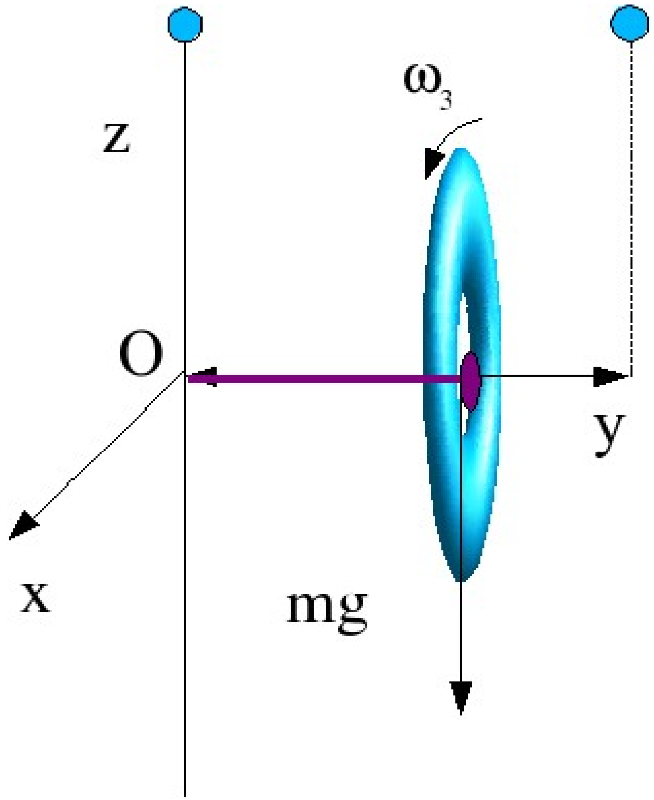
$$M = M g l \sin\phi$$

$$(I_b + M L^2)\ddot{\phi} = + M g l \sin\phi$$

Se attrito su x .. quando sta ferma?

Corpi rigidi

Rotore



$$\vec{L} \equiv (lx, ly, lz)$$

$$\vec{\omega} \equiv (\omega_x, \omega_y, \omega_z)$$

$$inizialmente\ \vec{\omega} \equiv (0, \omega_3, 0)$$

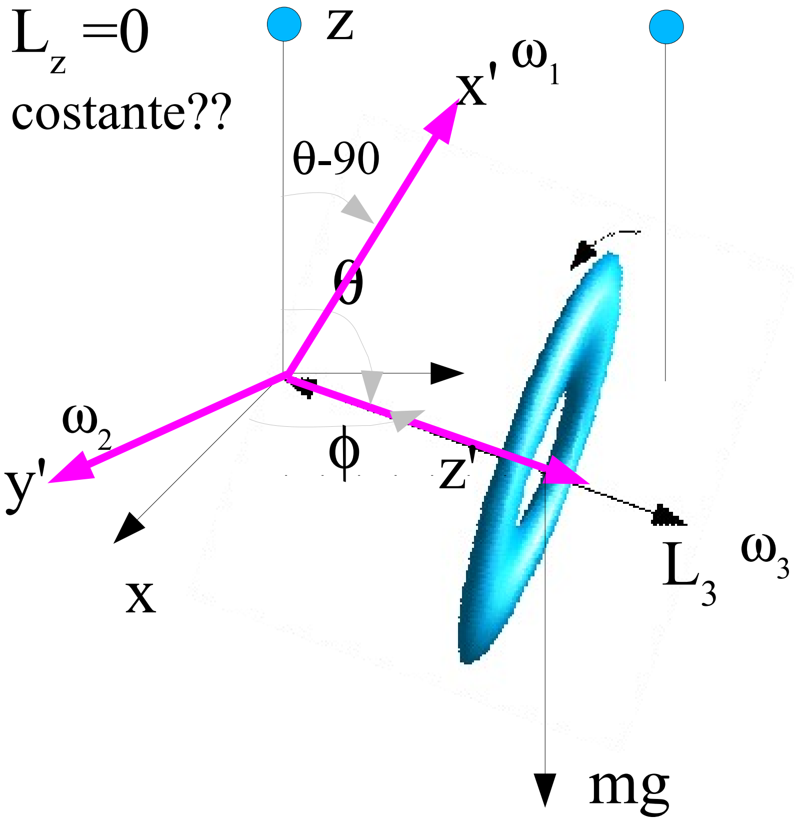
$$\vec{M} = \vec{r} \wedge m\vec{g} \quad con \quad M_z = 0$$

Corpi rigidi

Rotore

$L_z = 0$

costante??



$\omega_1 = \vec{\omega} \cdot \hat{x}' \quad \omega_2 = \vec{\omega} \cdot \hat{y}' \quad \omega_3 = \vec{\omega} \cdot \hat{z}'$

$L_1 = \vec{L} \cdot \hat{x}' = I_1 \omega_1$

$L_2 = \vec{L} \cdot \hat{y}' = I_2 \omega_2 \quad nota\ I_1 = I_2$

$L_3 = \vec{L} \cdot \hat{z}' = I_3 \omega_3$

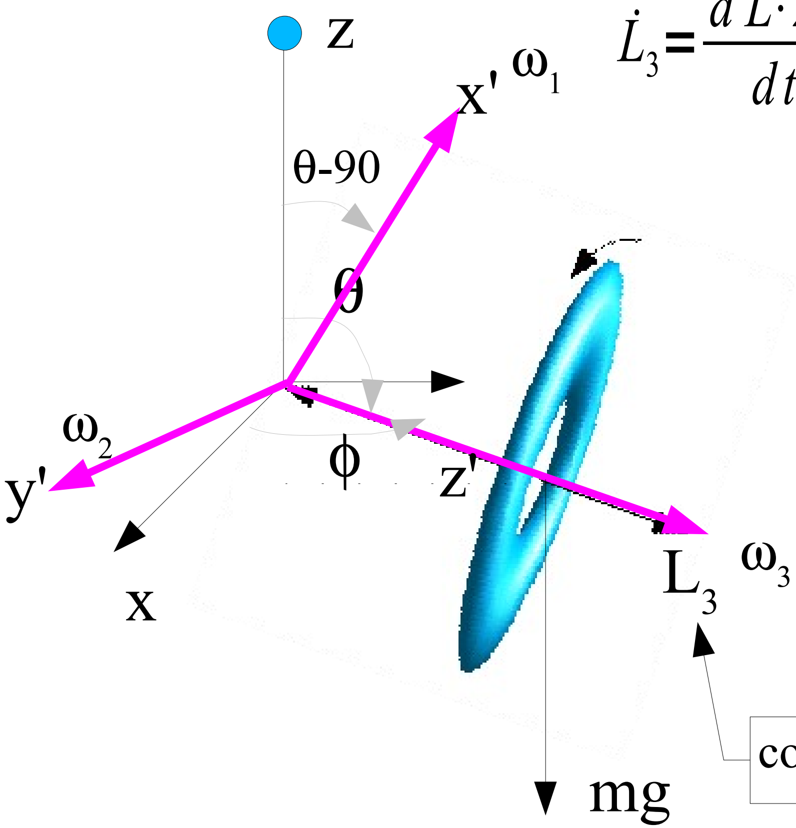
$\vec{M} = \vec{r} \wedge m \vec{g} \quad con \quad M_z = 0$

$\vec{M} \cdot \hat{z}' = M_3 = M_{z'} = 0$

$\vec{r} \| \hat{z}$

Corpi rigidi

Rotore



$$\dot{L}_3=\frac{d\vec{L}\cdot\hat{z}'}{dt}=\dot{\vec{L}}\cdot\hat{z}'+\vec{L}\cdot\dot{\hat{z}}'=\vec{M}\cdot\hat{z}'+\vec{L}\cdot(\vec{\omega}\wedge\hat{z}')=(\vec{L}\wedge\vec{\omega})\cdot\hat{z}'$$

$$\frac{d\vec{L}\cdot\hat{z}'}{dt}=(\vec{L}\wedge\vec{\omega})\cdot\hat{z}'=L_1\omega_2-L_2\omega_1=(I_1-I_2)\omega_1\omega_2=0$$

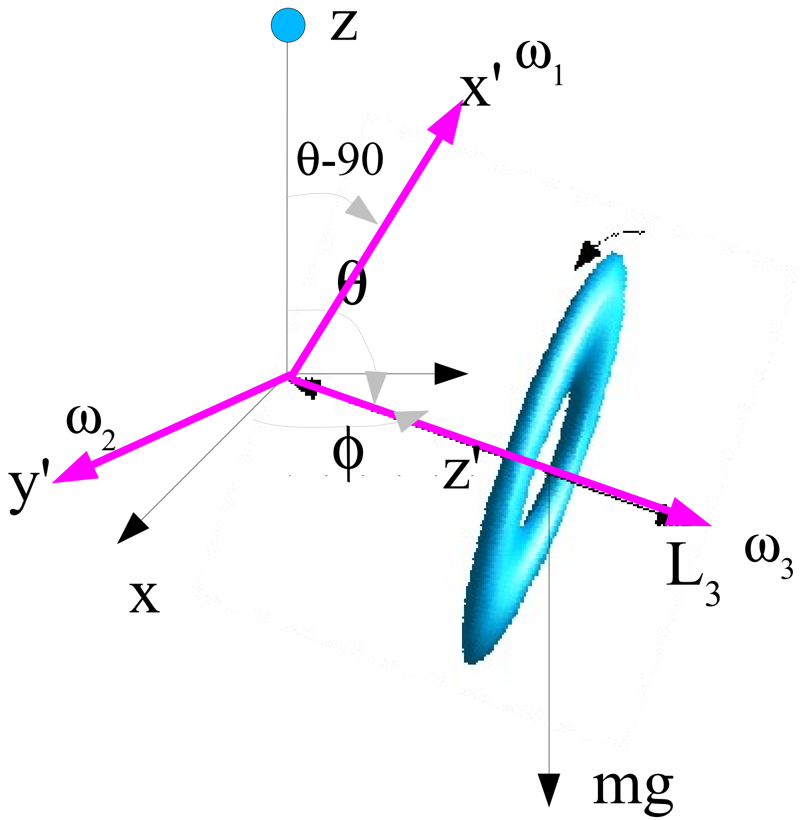
costante del moto

$L_z=0$

costante??

Corpi rigidi

Rotore



$$E=\frac{1}{2}\vec{L}\cdot\omega+mgd\cos\theta$$

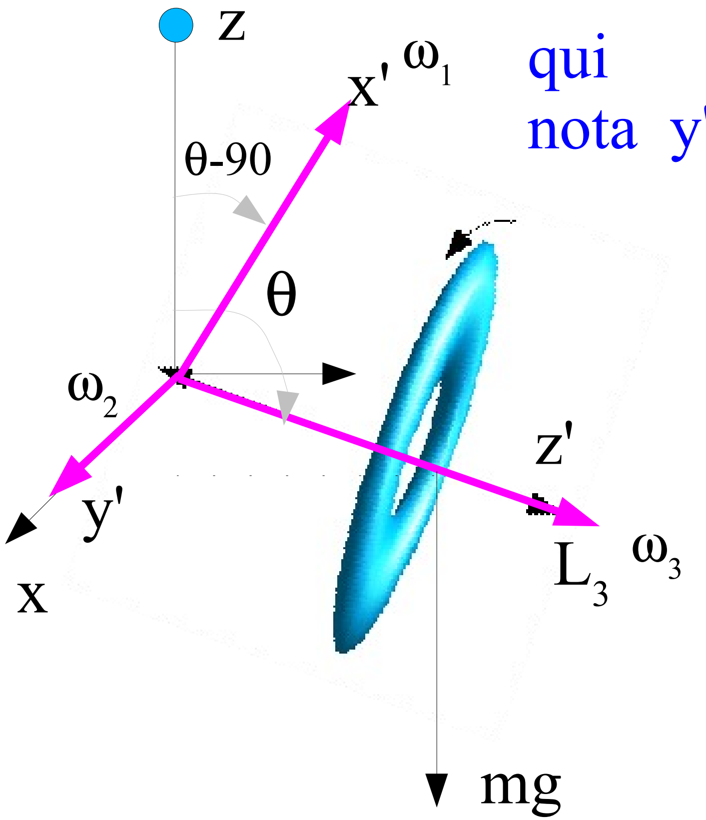
$$T=\frac{1}{2}\vec{L}\cdot\omega=\frac{1}{2}(L_1\omega_1+L_2\omega_2+L_3\omega_3)$$

$$=\frac{1}{2}(I(\omega_1^2+\omega_2^2)+I_3\omega_3^2)$$

$$=\frac{1}{2}I\omega_T^2+\frac{L_3^2}{2I_3}$$

Corpi rigidi

Rotore



qui
nota y' // x

$$\frac{1}{2}L_2\omega_2=\frac{1}{2}L_2\dot{\theta}_2=\frac{1}{2}I\dot{\theta}_2^2$$

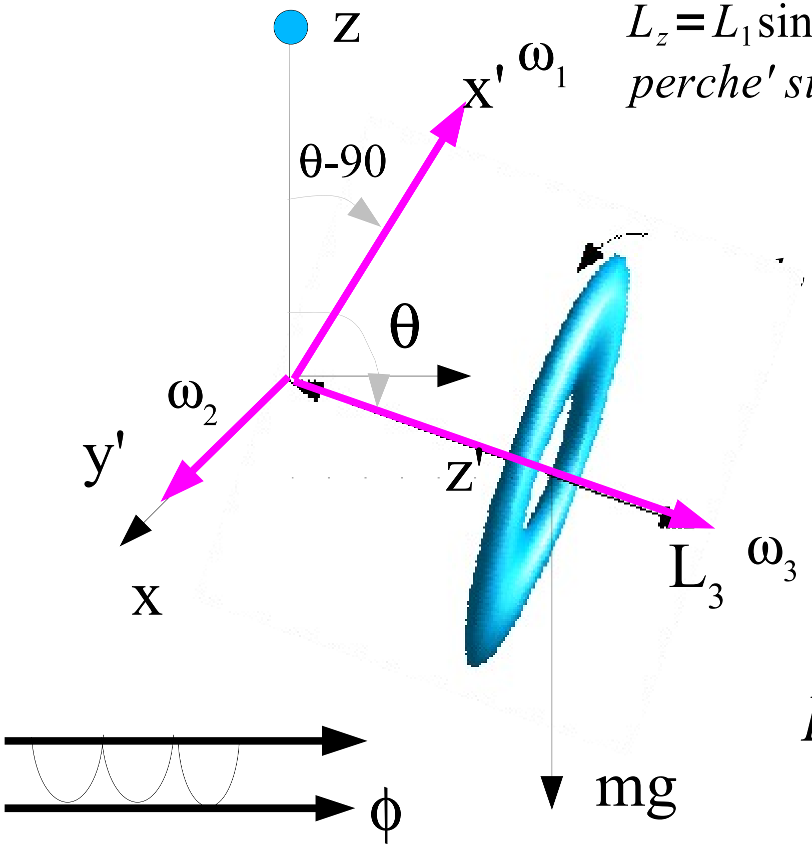
$$\frac{1}{2}L_1\omega_1=\frac{1}{2}I\omega_1^2=\frac{1}{2}I\sin^2\theta\,\dot{\phi}^2$$

$$E-\frac{L_3^2}{2I_3}=\frac{1}{2}I\omega_T^2+mgd\cos\theta$$

$$\frac{1}{2}I(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2)+mgd\cos\theta$$

Corpi rigidi

Rotore



$$L_z = L_1 \sin\theta + L_3 \cos\theta = I\,\omega_1 \sin\theta + L_3 \cos\theta = I \sin^2\theta\, \dot{\phi} + L_3 \cos\theta$$

perche' si e' scelto data la simmetria di rotazione $\hat{y}' \perp zz'$

$$\dot{\phi} = \frac{L_z - L_3 \cos\theta}{I \sin^2\theta}$$

che per le nostre condizioni iniziali Lz e' nulla

$$E' = E - \frac{L_3^2}{2 I_3}$$

$$E' = \frac{1}{2} I \dot{\theta}^2 - \frac{(L_3 \cos\theta)^2}{2 I \sin^2\theta} + m g d \cos\theta$$

U_{eff}