Pendolo semplice sul piano

$$m\ddot{\vec{r}} = m\vec{g} + \vec{R} \qquad ovvero$$

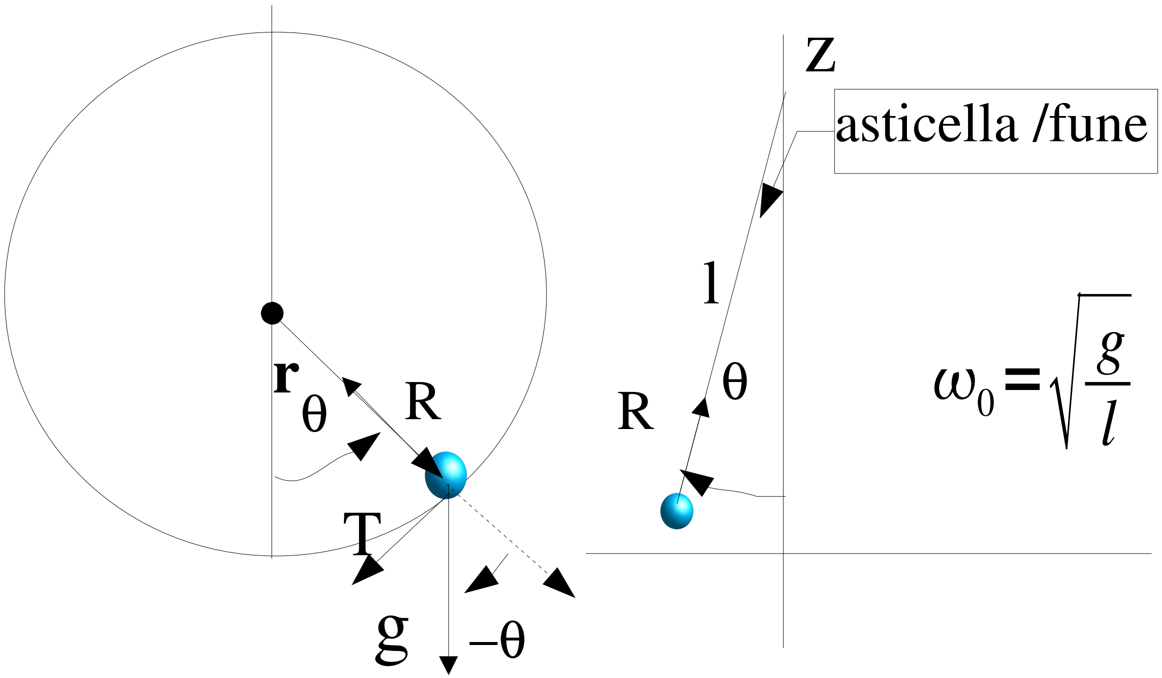
$$ml\ddot{\hat{r}} = m\vec{g} + \vec{R}$$

$$\ddot{\hat{r}} = \dot{\vec{\omega}} \wedge \hat{r} - \omega^2 \hat{r}$$

$$ml\ddot{\theta}\hat{T} - ml\dot{\theta}^2\hat{r} = m\vec{g} + \vec{R}$$

$$\ddot{\theta} = -\frac{g}{l} sen\theta$$

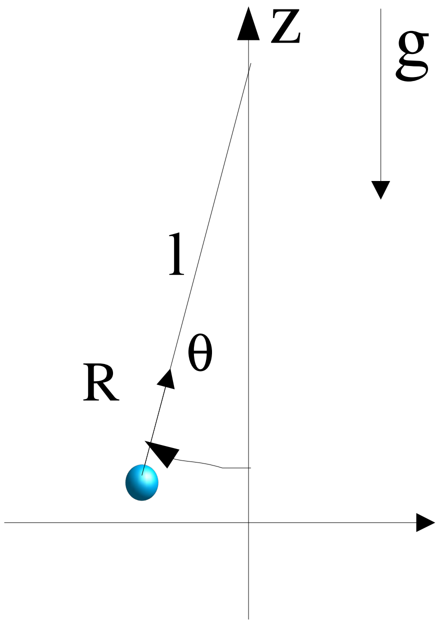
$$-\dot{\theta}^2 = \frac{g}{l}\cos\theta + \frac{R}{ml}$$



$$\omega_0 = \sqrt{\frac{g}{l}}$$

Pendolo
reazione vincolare

$$E=\frac{1}{2}mv^2+mgz=\frac{1}{2}m(l\dot{\theta})^2+mgl(1-\cos\theta)$$



$$\dot{\theta}^2=2\left(\frac{g}{l}\cos\theta+\frac{E}{ml^2}-\frac{g}{l}\right)$$

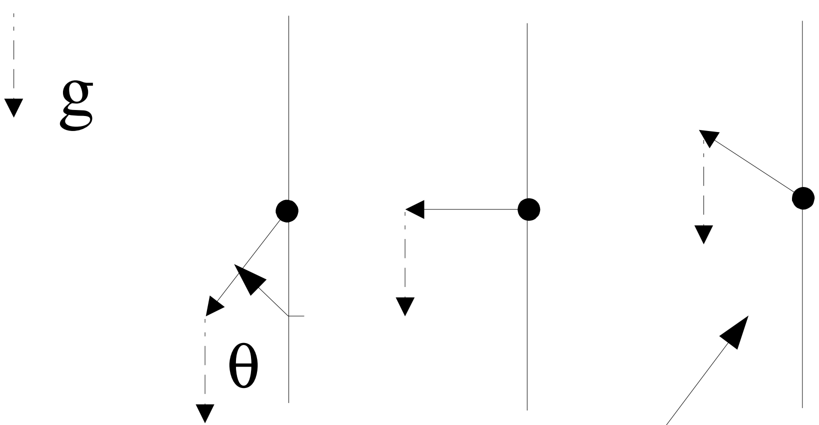
$$2\left(\frac{g}{l}\cos\theta+\frac{E}{ml^2}-\frac{g}{l}\right)=-\frac{g}{l}\cos\theta-\frac{R}{ml}$$

$$R=-2\frac{E}{l}-3\mathrm{mg}\cos\theta+2\mathrm{mg}$$

Pendolo

studio del moto da ω^2

$$\dot{\theta}^2 = 2\left(\frac{g}{l}\cos\theta + \frac{E}{ml^2} - \frac{g}{l}\right) = 0 \quad se \quad E = mgl(1-\cos\theta_{max}) \qquad R = -mg\cos\theta_{max}$$

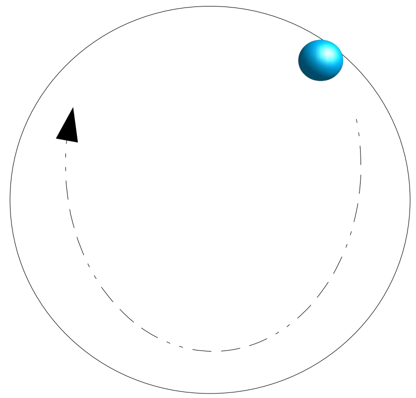


Se E < 2mgl la velocita' si annulla per un valore di θ minore di π. Il moto oscilla tra θ_{max} e $-\theta_{max}$

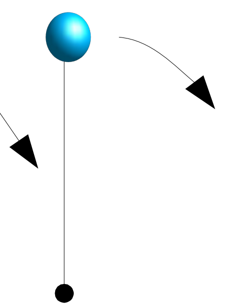
Se l'agolo massimo supera $\pi/2$ la reazione cambia segno e quindi non avremmo potuto usare un filo per fare il pendolo!

Pendolo
rotazione completa

$$E > 2mgl \quad segue$$
$$\dot{\theta}^2 > 0$$



Asticella

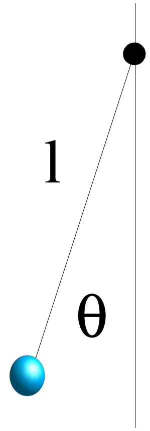


$$E = 2mgl \quad segue$$
$$\dot{\theta}^2 = 2\left(\frac{g}{l}\cos\theta + \frac{g}{l}\right) > 0$$

Pendolo

piccole oscillazioni $\ddot{\theta} = -\frac{g}{l} sen\theta$

$$\ddot{\theta} = -\omega_0^2 \theta \quad con \quad \omega_0^2 = \sqrt{\frac{g}{l}}$$



$$\sin\theta_{max} = \theta_{max} - \frac{1}{6}\theta_{max}^3 + ... = \theta_{max}\left(1 - \frac{1}{6}\theta_{max}^2\right) + ...$$

$$\frac{\sin\theta_{max} - \theta_{max}}{\theta_{max}} \approx \frac{1}{6}\theta_{max}^2 \quad segue \quad \theta_{max} \leq 0.2 \div 0.4 \quad all'1\ per\ cent$$

$$\tau = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}} \qquad segueper\ \ \tau = 1 \qquad l = \frac{g}{4\pi^2} \approx 25cm$$

Pendolo
in fluido viscoso

$$m\,l\,\ddot{\theta} = -m\,g\,\theta - \gamma\,l\,\dot{\theta} \qquad \ddot{\theta} = -\omega_0^2\theta - \frac{1}{\tau}\dot{\theta} \quad con \quad \tau = \frac{m}{\gamma}$$

$$m\,\ddot{x} = -k\,x - \gamma\,\dot{x}$$

$$\ddot{x} = -\omega_0^2\,x - \frac{\dot{x}}{\tau}$$

$$x = A\,e^{-\alpha t}\cos(\omega t - \phi)$$

Forze vive

$$m\,\ddot{x}\,\dot{x} = -k\,x\,\dot{x} - \gamma\,\dot{x}^2 \quad dacui$$

$$\frac{d}{dt}\left(\frac{1}{2}m v^2 + \frac{1}{2}k\,x^2\right) = -\gamma\,v^2 \quad ovvero$$

$$dE = -\gamma\,v \quad v\,dt = F_\tau\,v\,dt = F_\tau\,dx$$

energia dissipata in lavoro

Pendolo
soluzione del pendolo smorzato

$$\frac{d}{dt}e^{cx}=c\,e^{cx} \qquad e^{a+ib}=e^{a}e^{ib} \qquad e^{i\phi}=\cos\phi+i\sin\phi$$

$$x=A\,e^{\alpha t}$$

$$A\alpha^2 e^{\alpha t}=-A\omega_0^2 e^{\alpha t}-\frac{\alpha}{\tau}e^{\alpha t}$$

$$\alpha^2+\frac{\alpha}{\tau}+\omega_0^2=0$$

$$\alpha_+=-\frac{1}{2\tau}+i\omega \qquad \alpha_-=-\frac{1}{2\tau}-i\omega$$

$$\omega=\sqrt{\omega_0^2-\frac{1}{4\tau^2}}$$

$$e^{-\alpha_+ t}=e^{-\frac{t}{2\tau}}(\cos\omega t+i\sin\omega t)$$

$$e^{-\alpha_- t}=e^{-\frac{t}{2\tau}}(\cos\omega t-i\sin\omega t)$$

Pendolo

soluzione del Pendolo smorzato

$$\frac{d}{dt}e^{cx}=c\,e^{cx} \qquad e^{a+ib}=e^{a}e^{ib}$$

$$e^{-\alpha_+ t}=e^{-\frac{t}{2\tau}}\left(\cos\omega t+i\sin\omega t\right)$$

$$e^{-\alpha_- t}=e^{-\frac{t}{2\tau}}\left(\cos\omega t-i\sin\omega t\right)$$

$$x_1=\frac{e^{-\alpha_+ t}+e^{-\alpha_- t}}{2}=e^{-\frac{t}{2}\tau}\cos\omega t$$

$$x_2=\frac{e^{-\alpha_+ t}-e^{-\alpha_- t}}{2i}=e^{-\frac{t}{2}\tau}\sin\omega t$$

$$x=c_1x_1+c_2x_2=e^{-\frac{t}{2\tau}}\left(c_1\cos\omega t+c_2\sin\omega t\right)$$

Pendolo
pendolo smorzato soluzione generale

$$x = e^{-\frac{t}{2\tau}}\left(c_1 \cos\omega t + c_2 \sin\omega t\right) = e^{-\frac{t}{2\tau}} A\cos(\omega t + \phi)$$

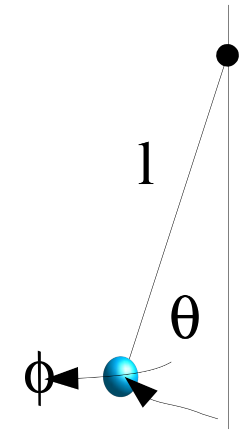
$$A = \sqrt{c_1^2 + c_2^2} \qquad \phi = \arctan\frac{c_1}{c_2} \qquad \omega = \sqrt{\omega_0^2 - \frac{1}{4\tau^2}}$$

- ω e' la nuova frequenza; diminuisce a causa dell'attrito
- se $\omega_0 < 1/2\tau$ pendolo sovrasmorzato ==> 2 exp decrescenti!
- se $\omega_0 > 1/2\tau$ il pendolo oscilla ma decresce la sua ampiezza
- t definisce il periodo di smorzamento; dopo 2τ l'ampiezza si e' ridotta di 1/e
- $Q = \omega_0\tau = 2\pi\tau/T_0$ definisce la bonta' dell'oscillatore come 2π volte il numero di oscillazioni che avvengono nel tempo caratteristico di smorzamento.

Pendolo
sferico

$$x = l\sin\theta\cos\phi$$
$$y = l\sin\theta\sin\phi$$
$$z = \cos\theta$$

$$v_x = l\dot{\theta}\cos\theta\cos\phi - l\dot{\phi}\sin\theta\sin\phi$$
$$v_y = l\dot{\theta}\cos\theta\sin\phi + l\dot{\phi}\sin\theta\cos\phi$$
$$v_z = -l\dot{\theta}\sin\theta$$
$$v^2 \equiv v_x^2 + v_y^2 + v_z^2 = l^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$$



$$E = \frac{1}{2}mv^2 + mgz = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mgl(1-\cos\theta)$$

$$L_z = m(xv_y - yv_x) = ml^2\dot{\phi}^2\sin^2\theta \quad \textit{sostituendo}\, L_z$$

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{l_z^2}{2ml^2\sin^2\theta} + mgl(1-\cos\theta)$$

$$U_{eff} = \frac{l_z^2}{2ml^2\sin^2\theta} + mgl(1-\cos\theta)$$

Pendolo
sferico

$$\frac{d}{dt}E=0 \qquad ml\ddot{\theta}=\frac{L_z^2\cos\theta}{ml^3\sin\theta^3}-mg\sin\theta$$

repulsiva

$$\frac{d}{dt}L_z \qquad \ddot{\phi}=\dot{\phi}\,\dot{\theta}\,tang\,\theta$$

$$U_{eff}=\frac{l_z^2}{2ml^2\sin^2\theta}+mgl(1-\cos\theta)$$

Bariera centrifuga

SE $L_z=ml^2\dot{\phi}^2\sin^2\theta=0$

$segue\ il\ moto \in un\ piano$